Ch.3 Numerical (iterative) solution of nonlinear equations

The formulation of a nonlinear problem in FEM leads to a system of nonlinear equations whose unknown values represent the numerical values of the sought functions at the nodes of the computational model. The total number of equations is equal to the total number of degrees of freedom in the model. To solve the system of nonlinear equations FEM programs use algorithms in the form of programmed solvers based on basic mathematical methods for solving such problems. Most often they use an iterative approach based on the Newton-Raphson method which we now remind with a special focus on the equations of structural problems.

Consider a bar of elastic material with Young's modulus E_0 and a cross-sectional area S_0 fixed and loaded as shown in Fig. 3.1. The bar has only one degree of freedom, that is, vertical displacement *u*, whose

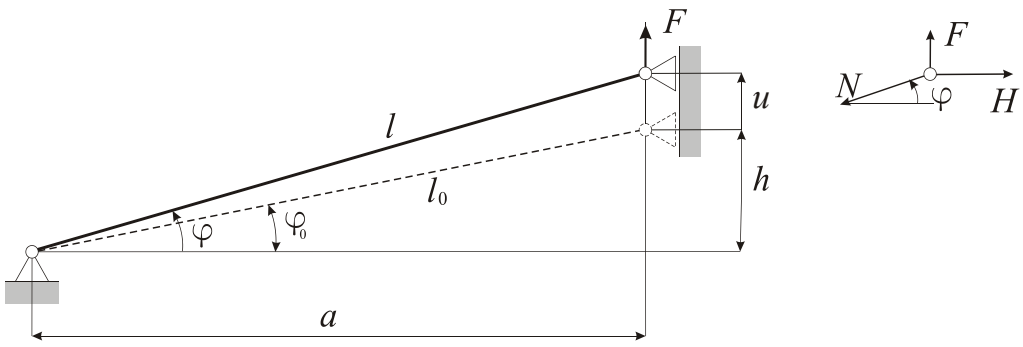

Fig. 3.1 Simple nonlinear problem

dependence on the load force *F* can be approximately expressed by a simple nonlinear equation [1]

$$\frac{E_0 S_0}{2l_0^3}\left(u^3 + 3hu^2 + 2h^2u\right) = F \tag{3.1}$$

With numerical values E_0 = 200 000 MPa, S_0 = 800 mm^2, l_0 = 2000 mm, h = 50 mm, equation (3.1) changes to

$$0{,}01\left(u^3 + 150u^2 + 5000u\right) = F \tag{3.2}$$

If we want to find the displacement *u* for a certain force value *F* iteratively, it is advantageous to write equation (3.2) in a form that makes it easy to control the iterative process

$$R(u) = 0{,}01\left(u^3 + 150u^2 + 5000u\right) - F = 0 \tag{3.3}$$

where *R(u)* is called the *non-equilibrium force*, because the equilibrium equation of the bar (3.2) is fulfilled only when this force equals zero. If, for example, for the calculation *u* we choose force *F* = 4000 N, the non-equilibrium force will be

$$R(u) = 0{,}01\left(u^3 + 150u^2 + 5000u\right) - 4000 \tag{3.4}$$

A graphical representation of this function in Fig. 3.2 shows that the displacement of the bar at *F* = 4000 N is approximately u^* = 35 mm.

```
F = 4000;
R[u_] = 0.01 (u^3 + 150 u^2 + 5000 u) - F;
Plot[{R[u]}, {u, 0, 80}, AxesLabel → {"u", "R(u)"},
  PlotPoints → 100, PlotStyle → {Thickness[0.003]},
  Ticks → {Automatic, {-4000, 5000, 10000, 15000}}];
```

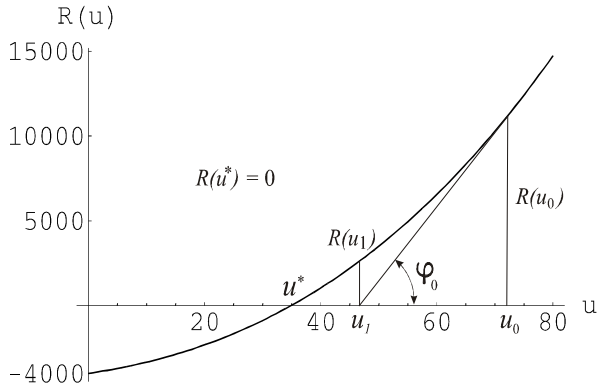

Fig. 3.2

The principle of the Newton-Raphson method can be easily explained by an equation with one unknown. We choose the initial value u_0 of the displacement (Fig. 3.2) and express the non-equilibrium force $R(u_0)$. At this point a tangent to the curve is made (this can be called linearization of the function around this point), which marks the displacement u_1 on the *u*-axis as the first approximation to the solution. This displacement is easy to determine because, as you can see from the picture, it applies

$$tg\left(\varphi_0\right) = \left(dR/du\right)_{u_0} = \frac{R\left(u_0\right)}{u_0 - u_1} \tag{3.5}$$

and from that

$$u_1 = u_0 - \frac{R\left(u_0\right)}{\left(dR/du\right)_{u_0}} = u_0 - \frac{R\left(u_0\right)}{\left(K_T\right)_{u_0}} \tag{3.6}$$

An analogous repetition of this procedure with values u_1, u_2 and so on, is the basis for the necessary iterative algorithm. The derivative

$$K_T\left(u\right) = \frac{dR\left(u\right)}{du} \tag{3.7}$$

is the so-called *tangential stiffness* which, as can be seen, is not constant in this process.

In the Newton-Raphson method, the root (solution) of a one-dimensional equation is searched as follows:

1. The gradient of the function *R* (*u*), the so-called tangential stiffness, is calculated

$$K_T\left(u\right) = \frac{dR\left(u\right)}{du} \tag{3.8}$$

2. The initial displacement u_0 is selected

3. In the cycle from *i = 0* to the selected value of the maximum allowable number of iterations $i_{\max}$

a) the following displacement value is calculated

$$u_{i+1} = u_i - \frac{R(u_i)}{K_T(u_i)} \tag{3.9}$$

b) the absolute value of the non-equilibrium force $|R_{i+1}| = |R(u_{i+1})|$ is compared with the prescribed tolerance R_{tol} and the decision-making steps are taken

- if $|R_{i+1}| \geq R_{tol}$, go to 3, if not, finish the solution
- if $i > i_{max}$, finish the solution

Example 3.1

For the bar fastened and loaded according to Fig. 3.3, the vertical displacement of the end point of the bar u_A is calculated using the Newton-Raphson method with a permissible solution error of R_{tol} = 1 N. Given: F = 4000 N, l_0 = 2000 mm, h = 50 mm, E_0 = 200 000 MPa, and S_0 = 800 mm^2.

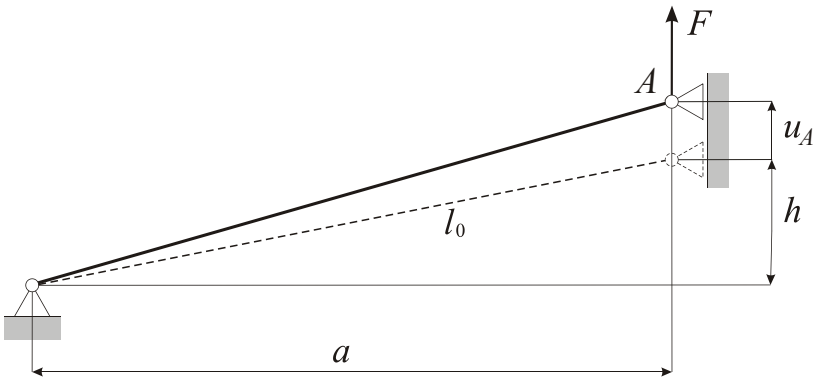

Fig. 3.3

The relationship between the displacement of the bar end point and the force F in the form of an non-equilibrium force is given by Equation (3.4)

$$R(u) = 0,01\left(u^3 + 150u^2 + 5000u\right) - 4000$$

with tangential stiffness $K_T(u) = \frac{dR(u)}{du} = 0,01\left(3u^2 + 300u + 5000\right)$

The first step of the iterative solution: (i = 0):

The initial displacement (selected) $u_0 = 0$

The first displacement value is $u_1 = u_0 - \frac{R(0)}{K_T(0)} = 0 - \frac{-4000}{50} = 80 \text{ mm}$

The absolute value of the non-equilibrium force at this offset value was (3.4) $|R_1| =$ 14720 N

Second step (i = 1):

The corrected displacement value is $u_2 = u_1 - \frac{R(80)}{K_T(80)} = 80 - \frac{14720}{482} = 49,46 \text{ mm}$

The non-equilibrium force is $|R_2| = 3352.5$ N

The third step (i = 2)

The corrected displacement value is $u_3 = u_2 - \frac{R(49,46)}{K_T(49,46)} = 49,46 - \frac{3352,53}{271,77} = 37,12$ mm

The non-equilibrium force dropped to $|R_3| = 435,28$ N.

To calculate the next values we used the program shown in Fig. 3.4, which also shows the calculated values of the example in the next steps. A graphical illustration of the iterative solution process is presented in Fig. 3.5.

```
NR[F_, uzac_, imax_, tol_] :=
 Module[{},
  r[u_] = 0.01 * (u^3 + 150 * u^2 + 5000 * u) - F;
  i = 0;
  u0 = uzac;
  Print[" u0 = ", u0, ", r0 = ", r[u0]];
  u1 = u0;
  While[i < imax && tol < Abs[r[u1]],
  u0 = u1;
  u1 = u0 - r[u0] / r'[u0];
  i = i + 1;
  Print[" i = ", i, ",   u(i) = ",
  PaddedForm[u1, {6, 4}], ", r(i) = ", PaddedForm [r[u1], {9, 4}]];
  ];]

NR[4000, 0, 7, 1];

u0 = 0, r0 = -4000

i = 1,   u(i) =  80.0000, r(i) =  14720.0000

i = 2,   u(i) =  49.4606, r(i) =   3352.5311

i = 3,   u(i) =  37.1248, r(i) =    435.2826

i = 4,   u(i) =  34.9776, r(i) =     11.9515

i = 5,   u(i) =  34.9152, r(i) =      0.0099
```

Fig. 3.4

From the result of the iteration it can be seen that a sufficiently accurate solution of equation (3.2) is the displacement with the value u_A = 34.92 mm.

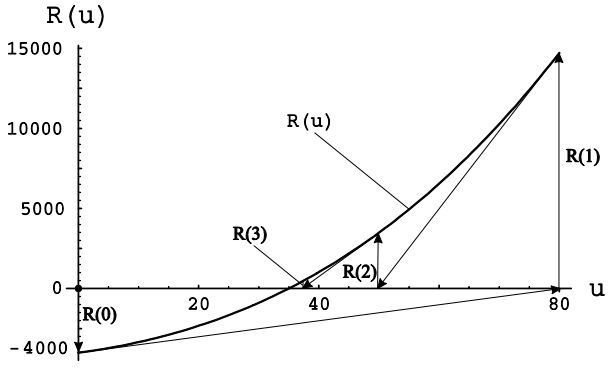

Fig. 3.5

For the general use of the method it is useful to recall its mathematical principle: the function $r(u)$ can be approximated (linearized) around the point u_i by decomposition into a truncated Taylor series

$$r(u) \approx r(u_i) + \frac{df(u_i)}{dx}(u - u_i) \tag{3.10}$$

The solution of the equation can then be obtained by the successive iterative solution of the equation

$$\begin{aligned} r(u_i) + \frac{df(u_i)}{dx}\Delta u = 0 \qquad i = 0,1,2,... \\ u_{i+1} = u_i + \Delta u \end{aligned} \tag{3.11}$$

to the extent of accuracy specified by the selected convergence condition (the convergence is not guaranteed for any type of function *f*).

The Newton-Raphson method can also be generalized for a system of nonlinear equations with several unknowns. This modification is normally used in FEM because the application of this method to a nonlinear problem leads to a system of such equations. The generalized displacements of the nodal points arranged in vector of functions **u** = $\left[u_1, u_2, u_3, \ldots, u_n\right]^T$ are unknown. In matrix notation, the system of equations of body equilibrium has the following shape

$$\mathbf{r}(\mathbf{u}) = \mathbf{q}(\mathbf{u}) - \mathbf{f} = \mathbf{0} \tag{3.12}$$

where **f** is the known external nodal force vector of the body and $\mathbf{q}(\mathbf{u})$ is the internal nodal force vector of the FEM model calculated from finite element stresses containing nonlinear functions of unknowns $\mathbf{u}$. Equations (3.12) express that the body is in equilibrium when the internal nodal forces are equal to the external nodal forces at each node (in the nodes without external forces, the internal nodal forces are in equilibrium). Using FEM, we create a system (3.12) and then, based on the selected initial values, iteratively adjust the displacements so that the non-equilibrium forces in the vector drop to sufficiently small values according to a certain prescribed convergence condition.

The Newton-Raphson method for the system of nonlinear equations **r**($\mathbf{u}$) = **0**, analogous to the one-dimensional case, consists of linearizing the components of the vector function

$$\mathbf{r}(\mathbf{u}) = \left[f_1(\mathbf{u}), f_2(\mathbf{u}), ..., f_n(\mathbf{u})\right]^T$$

using their first differential and replacing the nonlinear system of equations with a linear system to correct the arguments by $\Delta\mathbf{u}$. By solving this linear system, we obtain a new iteration $\mathbf{u} + \Delta\mathbf{u}$ of the approximate solution. By repeating this procedure we find the approximate value of the solution **u** with the required tolerance of either for $\mathbf{u}$ or **r**.

By linearizing the system **r**($\mathbf{u}$) = **0** around the initial approximation $\mathbf{u}_0$ we get

$$\mathbf{r}_L(\mathbf{u}) = \mathbf{r}(\mathbf{u}_0) + \frac{\partial \mathbf{r}(\mathbf{u}_0)}{\partial \mathbf{u}}(\mathbf{u} - \mathbf{u}_0)$$

We are looking for the first approximation $\mathbf{u}_1$ such that $\mathbf{r}_L(\mathbf{u}_1) = \mathbf{0}$. Then it applies

$$\frac{\partial \mathbf{r}}{\partial \mathbf{u}}(\mathbf{u}_0)(\mathbf{u}_1 - \mathbf{u}_0) = -\mathbf{r}(\mathbf{u}_0) \tag{3.13}$$

After selecting the initial displacement values $\mathbf{u}_0$, (3.13) represents a linear system of equations with an unknown vector $\Delta\mathbf{u}_0 = \mathbf{u}_1 - \mathbf{u}_0$. If the matrix of this system is regular then its solution is

$$\Delta\mathbf{u}_0 = -\left[\mathbf{K}_T(\mathbf{u}_0)\right]^{-1}\mathbf{r}(\mathbf{u}_0)$$

with the so-called tangential body stiffness matrix (Jacobi matrix)

$$\mathbf{K}_T = \frac{\partial \mathbf{r}}{\partial \mathbf{u}} = \begin{bmatrix} \partial R_1/\partial u_1 & \partial R_1/\partial u_2 & \cdots & \partial R_1/\partial u_n \\ \partial R_2/\partial u_1 & \partial R_2/\partial u_2 & \cdots & \partial R_2/\partial u_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial R_n/\partial u_1 & \partial R_n/\partial u_2 & \cdots & \partial R_n/\partial u_n \end{bmatrix} \tag{3.14}$$

The desired approximation is then given by $\mathbf{u}_1 = \mathbf{u}_0 + \Delta\mathbf{u}_0$. By repeating this procedure, we obtain the relation for the sequence of approximations

$$\mathbf{u}_{i+1} = \mathbf{u}_1 - \left[\mathbf{K}_T(\mathbf{u}_i)\right]^{-1}\mathbf{r}\left(\mathbf{u}_i\right) \qquad \text{i = 0, 1, 2, ..., } i_{max} \tag{3.15}$$

If this sequence converges to the solution of the system of equations (3.12), the calculation is terminated by the condition for $\left\|\mathbf{u}_{i+1} - \mathbf{u}_i\right\|$ or $\left\|\mathbf{r}_{i+1} - \mathbf{r}_i\right\|$.

In cases where the external nodal forces in (3.12) are not dependent on displacements, the tangential stiffness matrix can be expressed from the internal nodal forces as follows:

$$\mathbf{K}_T = \frac{\partial \mathbf{r}}{\partial \mathbf{u}} = \frac{\partial \mathbf{q}}{\partial \mathbf{u}} \tag{3.16}$$

The procedure, as shown in (3.15), requires inversion of the tangential body stiffness matrix at each iteration step, which can be time-consuming for large problems. However, in this case, the quadratic convergence of the iterative process is guaranteed and we are talking about the *full Newton-Raphson method.* In FEM programs it is usually possible to choose the *modified Newton-Raphson method*, where the inversion of the tangential stiffness matrix is performed only once at the beginning of the iteration process and does not change in the next iteration steps. Thus, the same matrix was used in all iteration steps. The iterative process is accelerated, but the number of iterative steps increases as the rate of convergence decreases.

Equation (3.15) can be adjusted to

$$\Delta\mathbf{u}_{i+1} = \mathbf{u}_{i+1} - \mathbf{u}_i = -\left[\mathbf{K}_T(\mathbf{u}_i)\right]^{-1}\left(\mathbf{q}_i - \mathbf{f}\right) \tag{3.17}$$

and further multiplied from the left by $\mathbf{K}_T\left(\mathbf{u}_i\right)$ to the form in which it usually occurs in program manuals

$$\mathbf{K}_T\left(\mathbf{u}_i\right)\Delta\mathbf{u}_{i+1} = \mathbf{f} - \mathbf{q}_i \tag{3.18}$$

Suppose that the iterative procedure begins with an unloaded body. Then, the node displacements are zero ($\mathbf{u}_0 = \mathbf{0} \;\rightarrow\; \mathbf{K}_T = \mathbf{K}_{lin}$) and since the stresses are zero ($\mathbf{q} = \mathbf{0}$) the initial iteration step is performed with a linear system of equations

$$\mathbf{K}_{lin}\mathbf{u}_1 = \mathbf{f}$$

From the calculated linear solution $\mathbf{u}_1$, the deformations and stresses in the elements are determined, from which $\mathbf{q}_1$ and the vector of non-equilibrium nodal forces of the first iteration step $\Delta\mathbf{r}_1 = \mathbf{f} - \mathbf{q}_1$ are

determined. A tangential stiffness matrix is also created for the next iteration step. According to(3.17), the displacement corrections $\Delta\mathbf{u}_{i+1}$ are then calculated

$$\mathbf{u}_{i+1} = \mathbf{u}_i + \Delta\mathbf{u}_{i+1}$$

With the new displacements, the iteration process continued until the convergence conditions were met. In the case of a linear problem, convergence is achieved in the first iteration.

In the case where the body and its loading form a so-called conservative system (meaning that no energy is dissipated during loading and the forces are conservative), it is also possible to formulate the equilibrium equations based on the principle of minimum total potential energy of the body, for which the following holds

$$\Pi = U - W = minimum \tag{3.19}$$

where *U* is the strain energy of the body and *W* is the potential energy of the external forces, referenced to the state of the body before deformation (which implies its negative sign, and the products of the external forces with displacements are not multiplied by one-half). The vector of unbalanced forces can then be expressed as the gradient of the extremum of this energy with respect to the displacement vector of the body's nodal points

$$\mathbf{r} = \frac{\partial \Pi}{\partial \mathbf{u}} = \mathbf{0} \tag{3.20}$$

According to (3.20), (3.19), and (3.12) we then obtain

$$\mathbf{r} = \frac{\partial U}{\partial \mathbf{u}} - \frac{\partial W}{\partial \mathbf{u}} = \mathbf{q} - \mathbf{f} = \mathbf{0} \tag{3.21}$$

For the internal nodal forces, it follows that

$$\mathbf{q} = \frac{\partial U}{\partial \mathbf{u}} \tag{3.22}$$

Example 3.2

The application of FEM to solve a simple geometrically non-linear truss system with a single free nodal point with three degrees of freedom led to a system of three nodal equations of equilibrium

$$R_1(u_1,u_2,u_3) = \left(0,01u_1^2 + 50\right)u_1 + 1,5u_1u_2^2 - 1,5u_1u_3^2 - 600 = 0$$
$$R_2(u_1,u_2,u_3) = 1,5u_1^2u_2 + \left(0,01u_2^2 + 50\right)u_2 + 1,5u_2u_3^2 - 800 = 0$$
$$R_3(u_1,u_2,u_3) = -1,5u_1^2u_3 + 1,5u_2^2u_3 + \left(0,01u_3^2 + 50\right)u_3 - 500 = 0$$

in matrix notation

$$\mathbf{r}(\mathbf{u}) = \mathbf{q}(\mathbf{u}) - \mathbf{f} = \mathbf{0}$$

where vector $\mathbf{f}$ contains the orthogonal components of the external force acting at the node.
The nodal displacement components $\mathbf{u} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T$ are determined using the Newton-Raphson method.

The *Mathematica 5* solution is shown in Fig. 3.6, where a set of equations is given in part a), the program for solving the problem is given in part b), the program start command with the necessary input parameters is in part c), and part d) contains an iterative sequence of solutions with the resulting non-equilibrium force vector satisfying the convergence condition.

```
R1[{u1_, u2_, u3_}] = (0.01 u1^2 + 50) u1 + 1.5 u2^2 u1 - 1.5 u3^2 u1 - 600;
R2[{u1_, u2_, u3_}] = 1.5 u1^2 u2 + (0.01 u2^2 + 50) u2 + 1.5 u3^2 u2 - 800;
R3[{u1_, u2_, u3_}] = -1.5 u1^2 u3 + 1.5 u2^2 u3 + (0.01 u3^2 + 50) u3 - 500;
R[{u1_, u2_, u3_}] = {R1[{u1, u2, u3}], R2[{u1, u2, u3}], R3[{u1, u2, u3}]};

NewtonRaphson[tol_, max_] :=
  Module[{norma = 1, i = 0},
   u0 = {0., 0., 0.};
   Print["u0 = ", u0];
   K_T[{u1_, u2_, u3_}] = Transpose[{∂u1 R[{u1, u2, u3}], ∂u2 R[{u1, u2, u3}], ∂u3 R[{u1, u2, u3}]}];
   While[And[i < max, norma > tol],
    u1 = u0 - (Inverse[K_T[u0]]).R[u0];
    Print["u"i+1, " = ", u1];
    norma = Sqrt[(u1 - u0).(u1 - u0)];
    If[norma < tol, Print["r"i+1, " = ", MatrixForm[R[u1]]]];
    u0 = u1;
    i = i + 1;]];

NewtonRaphson[10^-3, 30];
```

a

b

c

```
u0 = {0., 0., 0.}
u1 = {12., 16., 10.}
u2 = {8.09783, 11.0354, 6.77938}
u3 = {5.70581, 8.60238, 4.85192}
u4 = {4.40778, 8.32121, 3.8068}
u5 = {3.8515, 8.92314, 3.32305}
u6 = {3.73247, 9.16435, 3.21805}
u7 = {3.72762, 9.17659, 3.21387}
u8 = {3.72761, 9.17661, 3.21386}
r8 =
      ( -1.52482×10^-9 )
      |  -1.14767×10^-9 |
      ( -1.4295×10^-9  )
```

d

Fig. 3.6

From the results it can be seen that iteration begins with zero displacement components ($\mathbf{u}_0 = \mathbf{0}$), the vector $\mathbf{u}_1$ contains a linear solution and finally the vector $\mathbf{u}_8$ provides the values of the rectangular displacement components in the node (in mm) that satisfy the convergence condition. Vector $\mathbf{r}_8$ contains the values of the node non-equilibrium force components at such displacement values, i. j. values that meet the convergence condition.

In principle, FEM programs use the Newton-Raphson method, does the Ansys program, which will be used in this study to solve simple exemplary nonlinear problems.

Example 3.3

For the bar fixed and loaded as shown in the figure Fig. 3.3 , ANSYS was used to calculate the vertical displacement of the bar end point u_A when F = 4000 N, l_0 = 2000 mm, h = 50 mm, E_0 = 200000 MPa, and S_0 = 800 mm^2. See attached *Video 1*.

The result u_A = 34.996 mm is in good agreement with the result in Example 2.1.