Ch.1 Relations between displacement and deformation.

1.1 Small deformations

Consider a deformable body in a Cartesian coordinate system and load it with a static system of external forces. Because the body is fixed (does not move as a rigid unit), under the action of forces the initial configuration (geometry) of the body changes, and the points of the body move to a new position; the body deforms.

Suppose we know the vector of displacement of body points (the FEM provides them on finite elements in approximative form)

$$\mathbf{u}(x,y,z) \equiv u_i(x,y,z) \equiv \begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix} \tag{1.1}$$

where the continuous functions u, v, and w are the displacement components of the general xyz point in the direction of the coordinate axes. We are interested in the relationship between these functions and the searched functions, which express the degree of deformation of the body at its points.

Let us cosider the differential ("infinite" small) volume element $dxdydz$ inside the unloaded body at the general point xyz (Fig. 1.1).

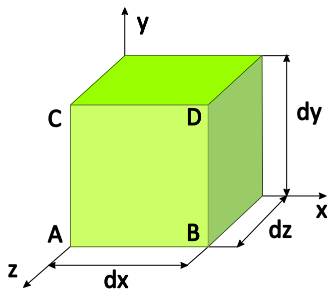

Fig. 1.1

Owing to the load, the element is displaced in space and deformed (because of its very small dimensions, a simple type of deformation related to the definition of stress is considered: its length dimensions change - in the direction of the coordinate axes stretches or shortens - and the right angles between its walls are violated).

Consider two points, A and B, on a parallel edge with the x-axis, spaced by dx (Fig. 1.2). The original positions of points A and B change as a result of the load. The line moves, changes its length, and rotates. For small deformations, we neglect the effect of the rotation on the length

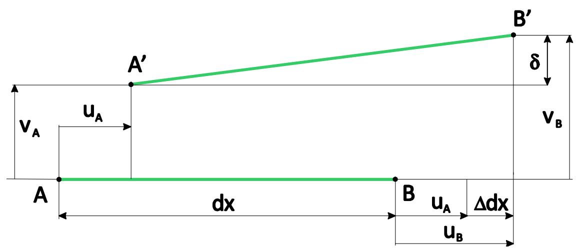

Fig. 1.2

of the line and for the degree of deformation (strain) of the element in the x- direction, the relative change in the x-distance of the points is chosen as

$$\varepsilon_x = \frac{\Delta dx}{dx} = \frac{u_B - u_A}{dx} \tag{1.2}$$

that is, the change in distance between points divided by the original distance, *dx*. Of course, if the element moves only as a whole ($u_B = u_A$) in the x-direction, then $\varepsilon_x = 0$. As shown above, the strain is a signed dimensionless number (with the physical meaning of relative extension or truncation).

The displacement u_B can be determined from the development of the function *u* at point A to the truncated Taylor series. Only the first two members are considered because of the small deformations

$$u_B = u_A + \frac{\partial u}{\partial x} dx \tag{1.3}$$

which represents the linearization of the function *u* at point A. By substituting (1.3) into (1.2), we obtain the relationship for calculating the strain ε_x at the points of the body:

$$\varepsilon_x(x,y,z) = \frac{\partial u}{\partial x} \tag{1.4}$$

It is actually the gradient of the function $u(x,y,z)$.

By an analogous analysis of the deformation of the element in the *y* and *z* directions, the following was obtained:

$$\varepsilon_y(x,y,z) = \frac{\partial v}{\partial y} \tag{1.5}$$

$$\varepsilon_z(x,y,z) = \frac{\partial w}{\partial z} \tag{1.6}$$

The functions, ε_x, ε_y and ε_z, known as *normal* strains, cause a change in the volume of the element without changing the angles between its walls because they only shift the parallel walls of the element by different values in the direction of their common normal.

However, the body load also causes so-called *shear* strains γ_{xy}, γ_{yz}, γ_{zx} which disrupt the perpendicularity of the walls but do not alter the volume of the element. For example, if point C (Fig. 1.1) shifts in the direction of the *x*-axis relative to point A by a value Δ, then, according to Fig. 1.3, by re-using the shortened Taylor series for the u_C we get

$$\Delta = u_C - u_A = u_A + \frac{\partial u}{\partial y} dy - u_A = \frac{\partial u}{\partial y} dy$$

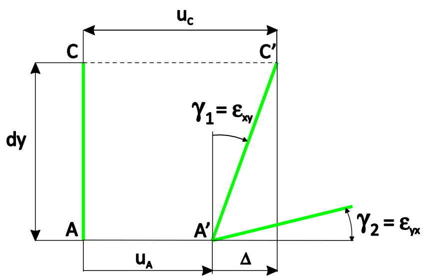

Fig. 1.3

In this way, the vertical wall of the element is tilted by an angle

$$\gamma_1 \approx tan\,\gamma_1 = \frac{\Delta}{dy} = \frac{\partial u}{\partial y}$$

Analogous to rotating the horizontal wall (Fig. 1.1) we have

$$\gamma_2 \approx tan\gamma_2 = \frac{\partial v}{\partial x}$$

and are the shear deformation in the xy plane (total change in the right angle of the element in the xy plane).

$$\gamma_{xy} = \gamma_1 + \gamma_2 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{1.7}$$

By cyclic interchange of variables we get shear deformations in the other two planes

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \tag{1.8}$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \tag{1.9}$$

In summary, the *linear* shear components at the general point of the body can be expressed as a (pseudo) vector:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y} \\ \dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial y} \end{Bmatrix} \tag{1.10}$$

The matrix notation of the symmetrical strain *tensor* considers the tensor shear deformation $\varepsilon_{ij} = \varepsilon_{ji}$ (Fig. 1.3) on each wall of the element (as a half value of the total shear deformation) and satisfies

$$\varepsilon = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_x & 0{,}5\gamma_{xy} & 0{,}5\gamma_{xz} \\ 0{,}5\gamma_{xy} & \varepsilon_y & 0{,}5\gamma_{yz} \\ 0{,}5\gamma_{xz} & 0{,}5\gamma_{yz} & \varepsilon_z \end{bmatrix} \tag{1.11}$$

where we get members from a simple index entry

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{1.12}$$

Using the deformation formulation of the strength problem, where the displacement field is primarily unknown, the strain relations (1.10) clearly determine the deformation field. The inverse problem that occurs in the force formulations is more complex because, in the case of six deformation components, the three displacement components can be unambiguously determined only with respect to the constraints contained in the compatibility equations.

1.2 Large strains

Linear geometric relations between displacement and strain components (1.10), resp. (1.12) at the material points of the deformed body are well suited for a large number of strength problems, since important materials used in engineering (metals, concrete, wood, glass, etc.) allow only small values of strains. For example, if a one-meter-long rod with a constant cross-section is stretched by 1 mm, it causes a numerically small strain. However, such a strain in the steel rod causes a stress of approximately 200 MPa, which is at the limit of the yield stress of conventional steel; for several other less ductile materials, it is already beyond the strength limit.

In contrast, several materials (e.g., industrial rubber and certain plastics) and technological processes (e.g., different types of steel forming) do not fit linear geometric equations, and more precise relationships must be used in simulation calculations. The derivation of these relationships is not easy; there are several methods and applied measures of deformation, and their application in FEM is not straightforward. Here, we present only a geometrical method for deriving the components of the Green–Lagrange strain tensor, which provides some insight into nonlinear geometric relations; however, it is necessary to state that the properties and possibilities of utilization of each measure of strain can only be more clearly explained together with its power partner - stress.

Returning to Fig. 1.2, the change in the length of the connecting line of points A and B is specified, considering the effect of different displacements of these points in the y-axis direction. We denote this difference as δ and by approximating the function around A by the shortened Taylor series:

$$v_B = v_A + \frac{\partial v}{\partial x} dx$$

so

$$\delta = v_B - v_A = \frac{\partial v}{\partial x} dx$$

and the deformed length of the line is expressed by the Pythagorean theorem

$$A'B' = \sqrt{\left(dx + \frac{\partial u}{\partial x} dx\right)^2 + \left(\frac{\partial v}{\partial x} dx\right)^2} = dx\sqrt{1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}$$

We introduce a relative deformation (marked as for small deformations)

$$\varepsilon_x = \frac{A'B' - AB}{AB} = \frac{A'B'}{dx} - 1 = \sqrt{1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2} - 1$$

The function of shape $\sqrt{1+x}$ can be decomposed into a series, and the previous relationship after approximating the member with a square root by two series members is given by

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] \tag{1.13}$$

Analogously, it can be shown that

$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \tag{1.14}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial x} \tag{1.15}$$

It can thus be observed that when the deformations are sufficiently small, the quadratic terms can be neglected, and the deformation components derived for the linear case are obtained.

Previous relations can be completed by analogous terms for *z*-direction, and we can write the *independent* components of the Green-Lagrange strain tensor as a vector

$$\varepsilon = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial z} \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\ \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x} \end{Bmatrix} = \begin{bmatrix} \frac{1}{2}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right) \\ \frac{1}{2}\left(\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right) \\ \frac{1}{2}\left(\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right) \\ \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) \\ \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial z}\right) + \left(\frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial z}\right) + \left(\frac{\partial w}{\partial y}\right)\left(\frac{\partial w}{\partial z}\right) \\ \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial z}\right) + \left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial v}{\partial z}\right) + \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial z}\right) \end{bmatrix} \tag{1.16}$$

or in a brief index tensor notation

$$\varepsilon_{k\ell} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_\ell} + \frac{\partial u_\ell}{\partial x_k} + \frac{\partial u_m}{\partial x_k}\frac{\partial u_m}{\partial x_\ell}\right) \tag{1.17}$$

Remark 1: If a bar (rod, truss) with a length ℓ_0 parallel to the x-axis at the end (located at the beginning of the coordinate system) is fixed and stretched by $\Delta\ell$ at the other end, its length will change to $\ell = \ell_0 + \Delta\ell$ and the displacement function will be

$$u(x) = \frac{\Delta\ell}{\ell_0}x = \frac{\ell - \ell_0}{\ell_0}x$$

According to relationship (1.4) valid for small deformations, the standard "engineering" strain of the bar can be expressed as follows:

$$\varepsilon_{ing} = \frac{\partial u}{\partial x} = \frac{\ell - \ell_0}{\ell_0}$$

From (1.13) we get the Green's strain

$$\varepsilon_G = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 = \varepsilon_{ing} + \frac{1}{2}\varepsilon_{ing}^2 = \frac{\ell - \ell_0}{\ell_0} + \frac{1}{2}\left(\frac{\ell - \ell_0}{\ell_0}\right)^2 = \frac{1}{2}\frac{\ell^2 - \ell_0^2}{\ell_0^2} \tag{1.18}$$

For small deformations ($\varepsilon_{ing} << 1$), the difference between the two measures is negligible:

$$\varepsilon_{ing} = \frac{l - l_0}{l_0} = \frac{(l - l_0)(l + l_0)}{l_0(l + l_0)} = \frac{l^2 - l_0^2}{l_0(l_0 + \Delta l + l_0)} = \frac{l^2 - l_0^2}{l_0^2(2 + \varepsilon_{ing})} \approx \frac{l^2 - l_0^2}{2l_0^2} = \varepsilon_G$$

The Green-Lagrange strain tensor is independent of the rigid displacement and solid rotation of the body and is therefore often used in the FEM to solve problems with small deformations but large displacements and rotations in the so-called corotational coordinate system.

Remark 2: We return to the figure from which the components of the Green–Lagrange strain tensor were determined.

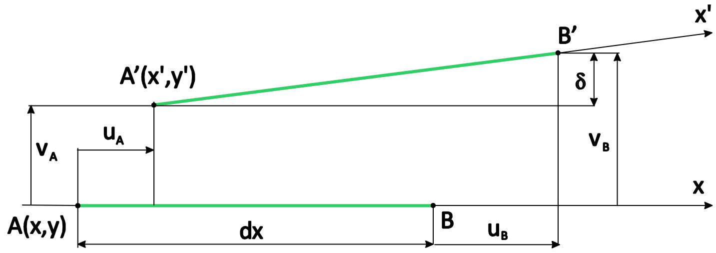

Fig. 1.4

The Green-Lagrange strain is, as already mentioned, expressed by the ratio

$$\varepsilon_x = \frac{A'B' - AB}{AB}$$

This is an expression analogous to that for small deformations; however, the directions of $A'B'$ and, consequently, are different from those of AB and dx, respectively. The component has a direction identical to that in which segment dx is rotated under the load. If in the next loading step, the line is only moved and rotated without changing the distance $A'B'$, the amount of deformation does not change. The components of the Green–Lagrange strain follow in their direction the rotation of the element (material particle), but their size is *independent* of this rotation (and of course, the rigid displacement). This is also valid for their partner (energy-bound) stress components of the second Piola–Kirchhoff stress.